Oriented Matroids Today

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Submitted: Apr 15, 2024; Accepted: Apr 15, 2024; Published: Apr 16, 2024

Abstract

This *dynamic survey*[1] provides three parts:

1. a sketch of a few "Frontiers of Research" in oriented matroid theory,
2. an update of corrections, comments and progress as compared to [BLVS+99], and
3. an extensive bibliography of oriented matroids, comprising and extending the bibliography of [BLVS+99]. (We believe this bibliography is complete up to 1993.)

Mathematics Subject Classifications: 52-00 (52B05, 52B30, 52B35, 52B40)

1 Introduction(s).

Oriented matroids were first motivated by abstracting geometric situations. To quote from [RGZ97]:

> The theory of oriented matroids provides a broad setting in which to model, describe, and analyze combinatorial properties of geometric configurations. Mathematical objects of study that appear to be disjoint and independent, such as point and vector configurations, hyperplane arrangements, convex polytopes, directed graphs, and linear programming find a common generalization in the language of oriented matroids.

Since that writing interest in oriented matroids has only expanded, both in their role in modeling geometric objects and as intriguing objects in ther own right. The main parts of the theory and some applications were compiled in 1993 in the comprehensive monograph

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[1] This is the fourth version. The previous versions were authored by Günter M. Ziegler.

 https://doi.org/10.37236/25

by Björner, Las Vergnas, Sturmfels, White & Ziegler [BLVS+99]. For other (shorter) introductions and surveys, see Bachem & Kern [BK92a], Bokowski & Sturmfels [BS89a], Bokowski [Bok93], Goodman & Pollack [GP93], Ziegler [Zie95, Chapters 6 and 7], and Richter-Gebert & Ziegler's handbook article [RGZ97], updated in [TGO17].

After a brief explanation of what an oriented matroid is, this *dynamic survey* provides three parts:

1. a sketch of a few "Frontiers of Research" in oriented matroid theory,
2. an update of corrections, comments and progress as compared to [BLVS+99], and
3. an extensive bibliography of oriented matroids, comprising and extending the bibliography of [BLVS+99]. (We believe this bibliography is complete up to 1993.)

2 What is an Oriented Matroid?

2.1 Motivation

Let $V = (v_1, v_2, \ldots, v_n)$ be a finite spanning sequence of vectors in $\mathbb{R}^r$, that is, a finite *vector configuration*. One can associate to V the following types of data, each of them encoding the *combinatorial structure* of V.

- The *chirotope* of V is the map

$$\begin{aligned} \chi_V : \{1,2,\ldots,n\}^r &\longrightarrow \{+,-,0\} \\ (i_1, i_2, \ldots, i_r) &\longmapsto \operatorname{sign}(\det(v_{i_1}, v_{i_2}, \ldots, v_{i_r})) \end{aligned}$$

  that records for each r-tuple of the vectors whether it forms a positively oriented basis of $\mathbb{R}^r$, a basis with negative orientation, or not a basis.

- The set of *covectors* of V is

$$\mathcal{V}^*(V) \quad := \quad \big\{\big(\operatorname{sign}(a^t v_1), \ldots, \operatorname{sign}(a^t v_n)\big) \in \{+,-,0\}^n : a \in \mathbb{R}^r\big\},$$

  that is, the set of all partitions of V (into three parts) induced by hyperplanes through the origin. It is also denoted as $\mathcal{L}$ sometimes, see below.

- The set of *signed cocircuits* of V is

$$\begin{aligned} \mathcal{C}^*(V) \;:=\; \big\{\big(\operatorname{sign}(a^t v_1), \ldots, \operatorname{sign}(a^t v_n)\big) \in \{+,-,0\}^n : a \in \mathbb{R}^n \text{ is orthogonal to a} \\ \text{hyperplane spanned by vectors in } V\big\}, \end{aligned}$$

  of all partitions by "special" hyperplanes that are spanned by vectors of the configuration V.

- The set of *vectors* of V is

$$\begin{aligned} \mathcal{V}(V) \;:=\; \big\{\big(\operatorname{sign}(\lambda_1), \ldots, \operatorname{sign}(\lambda_n)\big) \in \{+,-,0\}^n : \lambda_1 v_1 + \ldots + \lambda_n v_n = 0 \;\text{ is a} \\ \text{linear dependence between vectors in } V\big\}. \end{aligned}$$

- The set of *signed circuits* is

$$\mathcal{C}(V) \;:=\; \Big\{\big(\mathrm{sign}(\lambda_1),\ldots,\mathrm{sign}(\lambda_n)\big) \in \{+,-,0\}^n : \lambda_1 v_1 + \ldots + \lambda_n v_n = 0 \text{ is a } \textit{minimal} \text{ linear dependence between vectors in } V\Big\}.$$

It is not hard to see that all of these sets of data are equivalent, except for a global sign change that identifies χ with $-\chi$. That is, whenever one of the data

$$\{\chi_V, -\chi_V\}, \qquad \mathcal{V}^*(V), \qquad \mathcal{C}^*(V), \qquad \mathcal{V}(V), \qquad \text{or} \quad \mathcal{C}(V)$$

is given, one can from this uniquely reconstruct all the others.

Vector configurations as discussed above give rise to oriented matroids of *rank* r on n *elements* (or: on a *ground set* of size n). Usually the ground set is identified with $E = \{1, 2, \ldots, n\}$.

Equivalent to vector configurations, one has the model of (real, linear, essential, oriented) *hyperplane arrangements*: finite collections $\mathcal{A} := (H_1, H_2, \ldots, H_n)$ of hyperplanes (linear subspaces of codimension one) in $\mathbb{R}^r$, with the extra requirement that $H_1 \cap \ldots \cap H_n = \{0\}$, and with a choice of a positive halfspace H_i^+ for each of the hyperplanes. In fact, every vector configuration gives rise to such an arrangement via $H_i^+ := \{x \in \mathbb{R}^r : v_i^t x \geqslant 0\}$, and from an oriented hyperplane arrangement we recover a vector configuration by taking the positive unit normals. The set of covectors is the most interesting data set for this model. For each $x \in \mathbb{R}^r$ we define a vector $s(x) \in \{0, +, -\}^n$, where $s(x)_i$ is $+$ if x is on the positive side of H_i, $-$ if x is on the negative side of H_i, and 0 if $x \in H_i$. Then the range of s is the covector set of the oriented matroid associated to the arrangement.

Yet another route to oriented matroids comes via linear subspaces of $\mathbb{R}^n$. If V is such a subspace, then $\{\mathrm{sign}(x) : x \in V\}$ is the set of covectors of an oriented matroid. Another way to describe this oriented matroid is as arising from the oriented hyperplane arrangement in V whose elements are the intersections of the coordinate hyperplanes in $\mathbb{R}^n$ with V. Equivalently, the oriented matroid arises from the vector arrangement consisting of the orthogonal projections of the coordinate vectors onto V. Interpreting the covector set as $\{\mathrm{sign}(x) : x \in V\}$ suggests the outsize role of covectors in oriented matroid theory. For instance, consider the unit sphere in V and its decomposition into cells given by the coordinate hyperplanes; these cells correspond exactly to the nonzero covectors. This is one reason to use the notation $\mathcal{L}$ for the covector set: in the case of an oriented matroid arising from a subspace of $\mathbb{R}^n$, $\mathcal{L} \cup \{\hat{1}\}$ is clearly the face lattice of a regular cell decomposition of the unit sphere in that subspace, and even for general oriented matroids $\mathcal{L} \cup \{\hat{1}\}$ is the face lattice of a regular cell decomposition of a sphere. (This is one part of the *Topological Representation Theorem* described in Section 2.2.)

More specialized, one has the model of *directed graphs*: if $D = (V, A)$ is a finite directed graph (with vertex set $V = \{0, 1, 2, \ldots, r\}$ and arc set $A = \{a_1, \ldots, a_n\} \subseteq V^2$), then one has the obvious "directed circuits" in the digraph that give rise to circuits in the sense of sign vectors in $\mathcal{C}(V) \subseteq \{+, -, 0\}^n$, while directed cuts give rise to covectors, and minimal directed cuts give rise to cocircuits. Thus one obtains the oriented matroid of a digraph, which can also, equivalently, be constructed by associating with each arc (i, j) the vector $e_i - e_j \in \mathbb{R}^r$, where we take e_i to be the i-th coordinate vector in $\mathbb{R}^r$ for $i \geqslant 1$, and $e_0 := 0$.

2.2 Abstraction

The examples of the previous section motivate combinatorial definitions of abstract chirotopes, circuits, and vectors (cf. [BLVS+99] Chapter 2). As in the case of data arising from vector arrangements, these data types are equivalent: a pair $\pm\chi$ of rank r chirotopes determines a rank r signed circuit set, and so forth. Thus there are combinatorial structures, called **oriented matroids**, that can equivalently be given by any of these five different sets of data, and defined/characterized in terms of any of the five corresponding axiom systems. (The proofs for the equivalences between these data sets resp. axiom systems are not simple.)

Although the axiom systems of oriented matroids describe the data arising from vector configurations very well, it is not true that every oriented matroid corresponds to a real vector configuration. In other words, there are oriented matroids that are not *realizable*. This points to some basic theorems and a few of the fundamental problems in oriented matroid theory:

- The *Topological Representation Theorem* (see [BLVS+99, Chap. 5]) shows that, just as a real hyperplane arrangement in $\mathbb{R}^r$ can be represented by an arrangement of equators in S^{r-1}, every rank r oriented matroid can be represented by an arrangement of "pseudo-equators" in S^{r-1}. (The term used for these "pseudo-equators" is *pseudospheres*: note that the meaning of this term in oriented matroids is different from that in other fields.)

- There is no finite set of axioms that would characterize the oriented matroids that are representable by vector configurations. In fact, even for $r = 3$ there are oriented matroids on n elements that are *minimally non-realizable* for arbitrarily large n. See [Vam78] and Section 8.3 of [BLVS+99] for details.

- The *realization problem* is a difficult algorithmic task: for a given oriented matroid, to decide whether it is realizable, and possibly find a realization. This statement is a by-product of the constructions for the Universality Theorem for oriented matroids, see below.

2.3 Maps of oriented matroids

There are two notions of maps of oriented matroids.

We say there is a *strong map* from an oriented matroid M to an oriented matroid N if every covector of N is a covector of M. (In particular, M and N are oriented matroids on the same ground set E. This is a departure from ordinary matroid theory, which involves a set map from the elements of M to the elements of N: in oriented matroid theory we assume this map to be the identity.) In this situation we also say that N is a *quotient* of M. (In work of Las Vergnas quotients are also called *perspectives*.) For each of our geometric motivating objects, we have a motivation for strong maps:

- If M arises from a sequence $(v_1, \ldots, v_n)$ of vectors then the oriented matroid of the image $(f(v_1), \ldots, f(v_n))$ under a linear map is a quotient of M.
- If M arises from a sequence $(H_1, \ldots, H_n)$ of oriented hyperplanes in a vector space V and W is a linear subspace of V then the oriented matroid associated to the sequence $(H_1 \cap W, \ldots, H_n \cap W)$ of oriented hyperplanes in W is a quotient of M.
- If M arises from a subspace V of $\mathbb{R}^n$ and W is a linear subspace of V then the oriented matroid associated to W is a quotient of M.

We say there is a *weak map* from an oriented matroid M to an oriented matroid N if every covector of N is obtained from a covector of M by changing some (possibly none) nonzero components to 0. The interpretation of weak map in each of our motivating examples is about moving into more special position: if the geometric object associated to N can be perturbed slightly to give a geometric object associated to M then there is a weak map from M to N. This motivation leads to a combinatorial notion of *spaces of oriented matroids*: see Sections 3.2 and 3.3

The Topological Representation Theorem tells us that to each oriented matroid M there is an associated simplicial sphere, the order complex of the poset $V^*(M)\backslash\{0\}$. This gives a contravariant functor from the category of oriented matroids and strong maps to the category of simplicial spheres and simplicial maps. We can also associate to a weak map from M to N a map of simplicial spheres, but this association is not functorial [And01].

3 Some Frontiers of Research.

Among current areas of research are several deep problems of oriented matroid theory that were thought to be both hard and fundamental, and are now gradually turning out to be just that.

Here we give sketches and pointers to the literature for just a few topics. (The selection is very much biased. We plan to expand and update regularly. Your help and comments are essential for that.)

3.1 Realization spaces.

Mnëv's Universality Theorem of 1988 [Mnë88] states that every primary semialgebraic set defined over $\mathbb{Z}$ is "stably equivalent" to the realization space of some oriented matroid of rank 3. In other words, the semialgebraic sets of the form

$$\mathcal{R}(X) \quad := \quad \{Y \in \mathbb{R}^{3\times n} : \operatorname{sign}(\det(X_{i,j,k})) = \operatorname{sign}(\det(Y_{i,j,k})) \text{ for all } 1 \leqslant i < j < k \leqslant n\},$$

for real matrices $X \in \mathbb{R}^{3\times n}$, can be arbitrarily complicated, both in their topological and their arithmetic properties. The Universal Partition Theorem [Mnë91], announced in

1991, says that essentially every semialgebraic family appears in the stratification given by the determinant function on the (3×3)-minors of $(3 \times n)$-matrices.

These results are fundamental and far-reaching. For example, via oriented matroid (Gale) duality they imply universality theorems for d-polytopes with $d+4$ vertices (cf. Chapter 6 of [Zie95]).

Mnëv's original proof of Universality was simplified greatly by Shor [Sho91], see also Richter-Gebert [RG95b]. The Universal Partition Theorem got its first complete proof by Günzel [Gün96] (in a weakened form) and by Richter-Gebert [RG95b].

Stable equivalence is a special kind of homotopy equivalence that in some sense preserves algebraic complexity. There are several vague or incomplete definitions of stable equivalence in the literature: we give a complete definition here, with thanks to Boege [Boe] and Verkama [Ver23] for elucidating the issues. Let V and W be semialgebraic sets. A *rational equivalence* from V to W is a function $f : V \to W$ such that both f and f^{-1} are rational functions with rational coefficients. If $V \subseteq \mathbb{R}^{n+m}$ and $W \subseteq \mathbb{R}^n$ is the projection of V to its first n coordinates, we say that the projection $\pi : V \to W$ is a *stable projection* if the following conditions are satisfied.

1. There are finite sets of polynomials $\{\Phi_i : i \in I\}, \{\Psi_j : j \in J\} \subset \mathbb{Q}[w, v]$ such that each polynomial has degree 1 in variables v and
$$V = \{(w, v) : w \in W, \forall i \ \Phi_i(w, v) > 0, \forall j \ \Psi_j(w, v) = 0\}.$$

2. π is a homotopy equivalence

The condition that π is a homotopy equivalence is satisfied, for instance, if π has a global section σ. In this case V retracts to $\sigma(W)$ by a fibrewise straight-line homotopy. *Stable equivalence* is the equivalence relation on semialgebraic sets generated by rational equivalence and stable projection.

Both the statement and proof techniques of the Universality Theorem have led to universality theorems for other moduli spaces. Richter-Gebert has proved a Universality Theorem (and Universal Partition Theorem) for 4-dimensional polytopes, and related to this a non-Steinitz theorem for 3-spheres [RGZ95a, RGZ95b]. See [Gün98] for a second proof. Kapovich & Millson [KM02] proved a Universality Theorem for configuration spaces of planar polygons. (Kapovich & Millson trace the history of their result back to a universality theorem by Kempe [Kem75] from 1875!) Lafforgue [Laf03] proved a scheme-theoretic version of the oriented Universality, which was the basis for a wide-ranging paper by Vakil [Vak06] on “Murphy's Law in algebraic geometry”. Dobbins, Holmsen and Hubard [DHH17] generalized rank 3 chirotopes to a notion of order type of a family of convex bodies in the plane, and they showed that, for each fixed $k \geqslant 3$, every primary semialgebraic set is stably equivalent to the space of realizations of an order type by k-gons in the plane.

Here are some major challenges that remain in this area:

- To construct and understand the smallest oriented matroids with non-trivial realization spaces. The smallest *known* examples are Tsukamoto's [Tsu13] oriented matroids of rank 3 on 13 points with a disconnected realization space (Tsukamoto produces two such oriented matroids, differing in a single basis.) The smallest known uniform example is Suvorov's [Suv88] oriented matroid of rank 3 on 14 points, also with a disconnected realization space (see also [BLVS$^+$99, p. 365]). Another interesting example is Richter-Gebert's [RG96b] non-uniform Ω_{14}^{+} with the same parameters, which additionally has rational realizations and a non-realizable symmetry.

- To provide Universality Theorems for *simplicial* 4-dimensional polytopes. (The Bokowski-Ewald-Kleinschmidt polytope [BEK84] is still the only simplicial example known with a non-trivial realization space; see also Bokowski & Guedes de Oliveira [BGdO90].)
- What is the topology of the space of all pseudosphere realizations of a fixed oriented matroid? In ranks 1 and 2 this space coincides with the realization space, and in rank 3 Dobbins [Dob21] has shown this space to be contractible, via a difficult proof using methods specific to this rank.

3.2 Extension spaces and liftings

Consider an an oriented matroid M on elements E. A *nontrivial single-element extension* of M is an oriented matroid M' on elements $E \cup \{p\}$ of the same rank as M such that the deletion $M' \backslash p$ is M. A *nontrivial single-element lifting* of M is an oriented matroid M'' on elements $E \cup \{p\}$ such that p is a nonloop and the contraction M''/p is M.

Lifting and extensions are dual concepts: M' is a nontrivial single element extension of M exactly when $(M')^*$ is a nontrivial single-element lifting of M^*.

3.2.1 The Bohne-Dress Theorem

The Bohne-Dress Theorem, announced by Andreas Dress at the 1989 "Combinatorics and Geometry" Conference in Stockholm, provides a bijection between the zonotopal tilings of a fixed d-dimensional zonotope Z and the single-element liftings of the realizable oriented matroid associated with Z. This theorem turned out to be, at the same time,

- fundamental (see e. g. the connection to extension spaces of oriented matroids [SZ93]),
- "intuitively obvious" (just draw pictures!), and
- surprisingly hard to prove; see Bohne [Boh92a] and Richter-Gebert & Ziegler [RGZ94].

A substantially different proof of the Bohne-Dress theorem was given by Huber, Rambau & Santos [HRS00]. In particular, there are bijections

$$\left\{\begin{array}{c}\text{zonotopal tilings of}\\ \text{the zonotope } \mathcal{Z}(A)\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{subdivisions of the}\\ \text{Lawrence polytope } \Lambda(A)\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{extensions of the dual}\\ \text{oriented matroid } \mathcal{M}^*(A)\end{array}\right\}$$

The first bijection follows from the "Cayley trick", see Huber, Rambau & Santos [HRS00]. The second, more difficult one was already before established by Santos [San02, HRS00].

A separate, simpler proof for the case of rank 3 — pseudoline arrangements are in bijection with zonotopal tilings of a centrally symmetric $2n$-gon — is contained in the work by Felsner & Weil [FW01].

The Bohne-Dress theorem provides a connection to several other areas of study. On the one hand, the classification and enumeration of rhombic tilings of a hexagon relates to the theory of plane partitions and symmetric functions; see e.g. Elnitsky [Eln93], Edelman & Reiner [ER96a].

On the other hand, there is a definite need for a better understanding of zonotopal tilings of the entire plane (or of $\mathbb{R}^d$). Two different approaches have been started by Bohne [Boh92b, Kapitel 5] and by Crapo & Senechal [CS97], but no complete picture has emerged, yet. This is of interest, for example, in view of the mathematical problems posed by understanding quasiperiodic tilings and quasicrystals; see Senechal [Sen90, Sen95].

Motivated by ideas of Leclerc and Zelevinsky [LZ98], the Bohne-Dress Theorem has been specialized to a bijection between maximal by size *M-separated collections* and fine zonotopal tiling by Galashin and Postnikov [GP23].

3.2.2 The Extension Space Conjecture is false.

The *extension space* is the order complex of the poset $\mathcal{E}(M)$ of nontrivial single-element extensions of M by a nonloop, ordered by weak maps. Much research for many years focused on the *Extension Space Conjecture*, which stated that if M is realizable and rank r then this order complex is homotopy equivalent to an $(r-1)$-sphere. (A standard abuse of notation is to refer to the topology of a poset when we mean the topology of its order complex: thus the conjecture is stated as "$\mathcal{E}(M)$ is homotopy equivalent to an $(r-1)$-sphere".)

If we fix a realization of M as a vector arrangement, then it's not hard to see that the subposet of $\mathcal{E}(M)$ arising from extensions of this arrangement is homeomorphic to an $(r-1)$-sphere. It's also not hard to see that this subposet may not be all of $\mathcal{E}(M)$. Nonetheless, the Extension Space Conjecture "feels right", and Sturmfels and Ziegler [SZ93] proved the Extension Space Conjecture for oriented matroids of rank at most 3 or corank at most 2 in 1993. Only in 2016 was the general Extension Space Conjecture disproved by Liu [Liu20]. His proof is probabilistic: he gives a randomized process to produce a rank 3 vector arrangement $\widetilde{E_N}$ on $6N$ elements, and he proves that, for large enough N, with probability greater than 0, $\widetilde{E_N}$ contains a subconfiguration E such that the oriented matroid dual to the oriented matroid of E has disconnected extension space. (This oriented matroid with disconnected extension space has corank 3.)

Mnëv and Richter-Gebert [MRG93] gave examples of non-realizable rank 4 oriented matroids with disconnected extension space.

The Bohne-Dress Theorem allows us to identify $\mathcal{E}(M)$ with a Baues poset $\omega(C^n \to Z)$, where C^n is the cube whose dimension is the number of elements of M and Z is the zonotope given by a realization of M. Thus as a corollary of Liu's result we get a counterexample to "the Generalized Baues Conjecture for cubes".

It would be interesting to have a better understanding of the subposet of $\mathcal{E}(M)$ arising from a realization of M. For instance, do different realizations always correspond to isotopic subposets of $\mathcal{E}(M)$? It would also be interesting to have an explicit counterexample to the Extension Space Conjecture — not just an existence proof — and to know more about the smallest rank and corank in which counterexamples occur.

3.3 Combinatorial Grassmannians and flag posets

The consideration of *spaces of oriented matroids* brings several very different lines of thinking into a common topological framework. Given a set S of oriented matroids, we obtain a partial order on S by weak maps, and from this we obtain a topological space, the order complex ΔS (the simplicial complex given by chains in the partial order; see Björner [Bjö95]). This simplicial complex can be viewed as a combinatorial analog to a vector bundle. Just as a vector bundle represents a continuous parametrization of a set of vector spaces, this topological space can be viewed as a "continuous" parametrization of elements of S. Such spaces have arisen in several contexts.

- We've already seen the *extension space* $\Delta\mathcal{E}(M)$ of M.
- If S is the set $\mathrm{MacP}(r, n)$ of all rank r oriented matroids on a fixed set of n elements, this space is the *MacPhersonian.*
- If S is the set $\mathcal{G}(r, M)$ of all rank r quotients of a fixed oriented matroid M, this space is the *combinatorial Grassmannian.* (In fact, this example essentially encompasses the previous two: The extension space $\Delta\mathcal{E}(M)$ of a oriented matroid M is a double cover of $\Delta\mathcal{G}(\mathrm{rank}(M) - 1, M)$, while if M is the unique rank n oriented matroid on a fixed set of n elements, then $\mathcal{G}(r, M) = \mathrm{MacP}(r, n)$.)
- More generally, if $1 \leqslant r_1 < \cdots < r_k$ is a sequence of integers, an $(r_1, \ldots, r_k)$-*flag* of oriented matroids is a sequence $M_1 \leftarrow \cdots \leftarrow M_k$ of oriented matroids and strong maps in which each M_i has rank r_i. If $r_k \leqslant \mathrm{rank}(M)$ and S is the set $\mathcal{G}(r_1, \ldots, r_k, M)$ of all $(r_1, \ldots, r_k)$-flags $M_1 \leftarrow \cdots \leftarrow M_k \leftarrow M$ then this space is the $(r_1, \ldots, r_k)$-*flag space.*

The Topological Representation Theorem tells us that the combinatorial Grassmannian $\mathcal{G}(1, M)$ associated to a rank r oriented matroid M is homeomorphic to $\mathcal{G}(1, \mathbb{R}^r)$. (Here $\mathcal{G}(r_1, \ldots, r_k, \mathbb{R}^r)$ denotes the space of flags $V_1 \subset \cdots \subset V_k$ of linear subspaces of $\mathbb{R}^r$, with each V_i of dimension i.) When $k > 1$ then $\mathcal{G}(k, M)$ need not have the same dimension as $\mathcal{G}(k, \mathbb{R}^r)$. Babson [Bab94] showed that $\mathcal{G}(2, M)$ and $\mathcal{G}(1, 2, M)$ are homotopy equivalent to $\mathcal{G}(2, \mathbb{R}^r)$ and $\mathcal{G}(1, 2, \mathbb{R}^r)$. A realization of a rank r oriented matroid M leads to a continuous map of flag spaces $c : \mathcal{G}(r_1, \ldots, r_k, \mathbb{R}^r) \to \Delta\mathcal{G}(r_1, \ldots, r_k, M)$, well-defined up to homotopy. When $(r_1, \ldots, r_k) \in \{(1), (2), (1, 2)\}$ then this map is a homotopy equivalence. However, in general things are not so well-behaved: the examples in Section 3.2.2 also have disconnected $\mathcal{G}(r - 1, M)$. Thus there are rank 4 nonrealizable M and high-rank realizable M with disconnected combinatorial Grassmannians.

The main open conjecture is that the maps $c : \mathcal{G}(r, \mathbb{R}^n) \to \Delta\,\mathrm{MacP}(r, n)$ are homotopy equivalences. A paper asserting a proof of this conjecture was withdrawn in

2009 [Bis03], [Bis09]. There are substantial grounds for pessimism. Mnëv's Universality Theorem implies that for realizable $M \in \mathrm{MacP}(r,n)$ the inverse images under c can have arbitrarily complicated topology. Also, for large n almost all elements of $\mathrm{MacP}(r,n)$ are nonrealizable, and so the image of c is a small subset of $\Delta\,\mathrm{MacP}(r,n)$. However, substantial progress has been made on the first few homotopy groups of the MacPhersonian (Anderson [And98]), for mod 2 cohomology (Anderson & Davis [AD02]), and for $r = 3$ (Dobbins [Dob21]). Three related survey articles are Mnëv & Ziegler [MZ93], Anderson [And99a], and Reiner [Rei99].

The MacPhersonian and combinatorial flag spaces arise in MacPherson's theory of *combinatorial differential manifolds* and *matroid bundles* [Mac93], [And99a] in which oriented matroids serve as combinatorial analogs to real vector spaces. This analogy leads to an intriguing and useful interplay between topology and combinatorics. On the one hand, appropriate combinatorial adaptations of classical topological methods for real vector bundles prove that for realizable M^n the map $c : G(k,\mathbb{R}^n) \to \mathcal{G}(k,M^n)$ induces split surjections in mod 2 cohomology [AD02]. On the other hand, combinatorial methods can be applied to topology as well. Any real vector bundle over a triangulated base space can be "combinatorialized" into a matroid bundle [Mac93] [AD02], giving a combinatorial approach to the study of bundles. The most notable success in this direction has been Gel'fand & MacPherson's [GM92] combinatorial formula for the rational Pontrjagin classes of a triangulated differential manifold. The paper [GM92] omits many proofs: Abawonse and Anderson [AA] gave a more detailed account.

The topological problems discussed in this section have close connections to classical problems of oriented matroid theory, such as the following: Las Vergnas' conjectures that every oriented matroid has at least one mutation (simplicial tope) and that the set of uniform oriented matroids of rank r on a given finite set is connected under performing mutations, see Subsection 3.7.1. In fact, if these conjectures are false, then the "top level" of the MacPhersonian, given by all oriented matroids without circuits of size smaller than r and at most one circuit of size r, cannot be connected.

Another conjecture of Las Vergnas says that every strong map can be factored as an extension followed by a contraction. As a motivating example, if M arises from a hyperplane arrangement and N arises from the intersection of that arrangement with a subspace V, then an appropriate extension would be by a collection of hyperplanes whose intersection is V. Despite its plausibility, the conjecture is false: it was first disproved by Richter-Gebert [RG93d], and more recently Wu has found a realizable counterexample [Wu21a].

An equivalent formulation of Las Vergnas's conjecture is that every strong map $M \to N$ can be interpolated to a flag $M = M^r \to M^{r-1} \to \ldots \to M^s = N$, where each M^i has rank i. Thus the results of Richter-Gebert and Wu on Las Vergnas's conjecture are also results on flag posets: they show that the projection $\mathcal{G}(2,3,M) \to \mathcal{G}(2,M)$ need not be surjective. See [Wu21a] for further discussion.

3.4 Realization algorithms.

The realizability problem — given a "small" oriented matroid, find a realization or prove that none exists — is a key problem not only in oriented matroid theory, but also for various applications, such as the classification of "small" simplicial spheres into polytopal and non-polytopal ones (see Bokowski & Sturmfels [BS87b,BS89a], Altshuler, Bokowski & Steinberg [ABS80], Bokowski & Shemer [BS87a]). The universality theorems mentioned above tell us that the problem is hard: in fact, in terms of Complexity Theory it is just as hard as the "Existential Theory of the Reals (ETR)," the problem of solving general systems of algebraic equations and inequalities over the reals [Sho91]. While it is not known whether ETR over $\mathbb{Q}$ is at all algorithmically solvable (see Sturmfels [Stu87e]), there are algorithms available that (at least theoretically) solve ETR over the reals. In general one has NP⊆ETR⊆PSACE, i.e., these problems can be solved with polynomial space. More precisely, for ETR Basu, Pollack & Roy [BPR98] currently have the best result:

> Let $\mathcal{P} = \{P_1, \dots, P_s\}$ be a set of polynomials in $k < s$ variables each of degree at most d and each with coefficients in a subfield $K \subseteq \mathbb{R}$.
> There is an algorithm which finds a solution in each connected component of the solution set, for each sign condition on $P_1, \dots, P_s$, in at most $\binom{O(s)}{k} s\, d^{O(k)} = (s/k)^k s\, d^{O(k)}$ arithmetic operations in K.

However, until now this is mostly of theoretical value. See the Matoušek's expository paper [Mat14] for more on the topic. What can be done for specific, explicit, small examples? Given an oriented matroid, it seems that

- one efficient algorithm (in practice) currently available to *find a realization* (if one exists) is the iterative "rubber band" algorithm described in Pock [Poc91] for the rank 3 case. Further, Firsching [Fir17] demonstrated that current nonlinear optimization software can be used very efficiently to find realizations. Randomized methods can be used to find realizations, but it seems hard to employ them for nonuniform oriented matroids: see Fukuda, Miyata and Moriyama [FMM13].
- the most efficient criterion (in practice) currently available *to certify/verify non-realizability* is the "bi-quadratic final polynomials" algorithm of Bokowski & Richter-Gebert [BR90b] which uses solutions of an auxiliary linear program to construct final polynomials. Note that here *bi-quadratic* is essential, because by the above [BPR98], there always exists a final polynomial which however is computationally infeasible to find. Another method for finding certificates for non-realizability is based on semidefinite programming was analyzed by Miyata, Moriyama, and Imai [MMI09b, MMI09a] and also applied for some examples of higher rank, but seems to be dominated by bi-quadratic final polynomials: see [FMM13].

Neither of these is guaranteed to work: but still the combination of all three parts was good enough for a (still unpublished) complete classification of all 312,356 (unlabeled reorientation classes of) uniform oriented matroids of rank 3 on 10 points into realizable and non-realizable ones (Bokowski, Laffaille & Richter-Gebert [BK92b]). This was extended to $n = 11$ by Aichholzer, Aurenhammer, and Krasser [AAK02, AK07] using simulated annealing and bi-quadratic final polynomials. Indeed, all non-realizable uniform oriented matroids rank 3 on up to 11 elements can be shown to be such using bi-quadratic final polynomials. An explicit example of a non-realizable oriented matroid Ω_{14}^{-} without a bi-quadratic final polynomial was constructed by Richter-Gebert [RG96b], and one on 12 points has been announced by Scheucher. Fukuda, Miyata, and Moriyama [FMM13] combine the above methods with some ad hoc computations to enumerate the realizable (non-uniform) oriented matroids of rank 3 on 9 elements and of rank 4 on 10 elements. Another idea is to encode oriented matroids as solutions to a satisfiability problem and then use SAT solvers. The first use of this idea is due to Schewe [Sch10]. One way is to code the single elements extensions of an oriented matroids as solutions of a satisfiability problem. However, perhaps the most straightforward method is to encode the chirotope axioms as a satisfiability problem: see [Sch21] for the acyclic case. Finally, we mention the work of Finschi and Fukuda [FF03] enumerating simple (uniform) oriented matroids; the data (still being maintained) can be found on the *Homepage of Oriented Matroids*.

Enumeration of special oriented matroids called oriented matroid polytopes is usually the first step for the enumeration of polytopal spheres. Pfeifle [Pfe24] has used a POLYMAKE implementation of a search algorithm to disprove the realizability of a balanced 2-neighborly 3-sphere constructed by Zheng [Zhe20], a family of highly neighborly centrally symmetric spheres constructed by by Novik and Zheng [NZ24], and several combinatorial prismatoids introduced by Criado and Santos [CS17]. The algorithm looks for positive monomial combinations of Plücker polynomials which vanish in any realization, thus creating a contradiction to realizability. Further results concerning (neighborly) oriented matroid polytopes can also be found in [RGZ97, Chapter 6.4] and [MP15, Pad13].

A very closely related topic is that of Automatic Theorem Proving in (plane) geometry. In fact, the question of validity of a certain incidence theorem can be viewed as the realizability problem for (oriented or unoriented) matroids of the configuration. Richter-Gebert [RG91] and Wu [Wu94] here present two (distinct) views of the topic, both with many of its ramifications.

Closing this section let us note that the paper [Hui86] is a joke by Goodman and Pollack, a fake paper that never existed (and an algorithm as announced in the title does not exist). Indeed the last name of the author means “cheater” in Finnish.

3.5 Positroids

A surprising interplay between oriented matroids, physics, and algebra has emerged, in the form or *positroids*. A positroid, or *positively oriented matroid*, is an oriented matroid on $[n]$ with a chirotope whose value on every increasing sequence is nonnegative. In particular, positroids can be considered ordered oriented matroids. Ardila, Rincón, and Williams [ARW17] showed that all positroids are realizable, thus confirming a conjecture of da Silva [dS87a]. In fact, the realization space of a positively oriented matroid is always a ball. Boretsky, Eur and Williams [BEW22] strengthened the result of [ARW17] by showing that every flag consisting of positroids of consecutive ranks is positively realizable.

This remains open for general flags. Positroids also came up in relation to semigroup rings [Blu01], and an unpublished work of Postnikov [Pos06] gives descriptions in terms of decorated permutations and plabic graphs. The connection to physics comes by way of scattering amplitudes and the *amplituhedron* (cf. [AHBC+16]). For a detailed account of this rapidly emerging field, we refer the reader to Williams's excellent survey article [Wil].

3.6 The Tutte polynomial

The Tutte polynomial $T_{\underline{M}}(x, y)$ of an ordinary matroid $\underline{M}$ has additional context and interpretation if $\underline{M}$ is the underlying matroid of an oriented matroid M. Notably, $T_{\underline{M}}(2, 0)$ counts the number of regions in an arrangement representing M. When the ground set is linearly ordered, the central description is in terms of orientation-activities [LV84c], counting the number of smallest elements of circuits and cocircuits and yielding geometric interpretations for the coefficients of the Tutte polynomial. The question of relating these parameters to basis-activities of ordinary matroids was raised by Las Vergnas in [LV84c] and answered in a series of papers by Gioan and Las Vergnas on the so-called active bijection, initiated in [Gio02].

A central concept is that of active partition of the ground set associated with any ordered oriented matroid, yielding a canonical decomposition into bounded minors, and yielding activity classes which form a partition of the set of reorientations into boolean lattices. Enumerative counterparts yield Tutte polynomial interpretations, like evaluations counting various types of reorientations, or like a formula in terms of β-invariants of minors, or a 4-variable expansion formula, see [GLV18b, GLV19]. The active bijection of an ordered oriented matroid relates bases and reorientations and involves three levels: the bounded level where each bounded region is associated with its unique fully optimal basis, a concept that strengthens oriented matroid (linear) programming optimality [GLV09b];the central level where activity classes correspond to bases in an activity preserving and canonical way; and the refined level which yields various bijections, such as a bijection between regions and no-broken-circuit subsets [GLV18b].

If M is regular, then some evaluations of the Tutte polynomial count circuit-cocircuit reversal classes of reorientations [Gio08,GY19], which in particular can be interpreted if M is graphic [Gio07, CYZ08]. Bijections and interpretations involving these classes, Ehrhart theory and further objects were developed in [BBY19]. These bijections can be somehow extended to general oriented matroids in terms of extension-lifting constructions, where they can also be seen as a particular case of the bounded level of the active bijection [BSY19, BSY23].

One of the central conjectures related to the Tutte polynomial of matroids is the Merino-Welsh conjecture, stating that $\max(T_{\underline{M}}(0, 2), T_{\underline{M}}(2, 0)) \geqslant T_{\underline{M}}(1, 1)$, originally for graphic matroids only [MW99]. If $\underline{M}$ is orientable then $T_{\underline{M}}(0, 2)$ and $T_{\underline{M}}(2, 0)$ count the regions in arrangements representing M and M^*. There was hope that this could be used to approach the Merino-Welsh conjecture. However, the conjecture was disproved also for orientable matroids [BCCP23]: the counterexamples are large enough uniform matroids in which every element is doubled with a parallel element. In the graphic setting, the problem remains open for sparse graphs [Tho10].

See the survey [Gio22b] for further details, problems and references.

3.7 Special cells and associated graphs

3.7.1 Topes

In a simple oriented matroid M, the topes are the covectors without 0-entries. They correspond to maximal cells in a topological representation, and they determine M uniquely. Axiomatizations in terms of topes are known due to Handa [Han90] and da Silva [dS95]. Viewing the topes as a set of vertices of the hypercube allows for interesting links to set systems. For instance, the rank coincides with the VC-dimension, and questions about sample-compression can be studied. This extends to the broader setting of complexes of oriented matroids. Hence, we explain them in Subsection 3.8.1.

Simplicial cells and the mutation graph A maximal cell in an arrangement representing a rank r oriented matroid must have at least r sides: we call a maximal cell *simplicial* if it has exactly r sides. Las Vergnas' simplex conjecture [LV80a] says that every oriented matroid has a simplicial cell. Up to today the conjecture is known to hold for realizable oriented matroids [Sha79], oriented matroids of rank at most 3 [Lev26], small rank 4 oriented matroids [BR01], rank 4 uniform matroid polytopes [Miy20], and for Euclidean oriented matroids and oriented matroids that have a general-position extension yielding a Euclidean affine oriented matroid [Man82, Theorem 7]. Indeed, the latter is the largest class of oriented matroids (of unbounded rank) known to satisfy Las Vergnas' simplex conjecture. This led Mandel to the "wishful thinking conjecture" that all oriented matroids are of this type [Man82, Conjecture 8]. Mandel's conjecture has been disproved [KM23a] by showing that in such oriented matroids every element is incident to a simplicial cell. There are examples of uniform oriented matroids of rank 4 violating this property on 21 [RG93d], 17 [BR01], and 13 [TH04] elements, respectively.

If an oriented matroid has a simplicial cell, then one can understand combinatorially what it means to transform the arrangement by *pulling one pseudosphere bounding the simplicial cell over its opposite vertex.* It is equivalent to changing the sign of the chirotope corresponding to the cell; this operation is often called a *mutation.* Las Vergnas's simplex conjecture can be restated as saying that the *mutation graph* on all oriented matroids of fixed rank on elements E has no vertices of degree 0. Cordovil and Las Vergnas [RS88] conjectured further that for all r, n the mutation graph on the set of *uniform* oriented matroids on elements E and rank r is connected. This would imply that any two uniform oriented matroids of the same rank and on the same elements have topological representations that can be transformed into each other by a sequence of these "pulling" operations.

By Ringel's Homotopy Theorem [Rin56, Rin57] Cordovil-Las Vergnas's conjecture holds for rank at most 3. Also, in all ranks the induced subgraph defined by realizable uniform oriented matroids is connected [RS88]. The conjecture has been verified for $n \leqslant 9$ in [KM23a]. Based on polynomials associated to an oriented matroid, Lawrence studied properties of mutation sequences transforming one oriented matroid into another if they exist: see [Law00, Law09].

Complete cells A maximal cell is *complete* if it is bounded by all the pseudospheres in the arrangement. In [Rou91] Roudneff conjectured that an oriented matroid of rank r and $n \geqslant 2r-1$ elements has at most $2\sum_{i=0}^{r-3}\binom{n-1}{i}$ complete cells. This bound is attained by the *alternating matroid*, the dual of the point matroid of the cyclic polytope. It further has been asked if also for $r \leqslant n \leqslant 2r-2$ the alternating matroid maximizes the number of complete cells [MR15], which was computed in [FR01]. To prove Roudneff's conjecture for a given r it suffices to do so for $n = 2r-1$: see [Rou91]. The latter allowed the conjecture to be verified for rank at most 5: see [Ram99, HOKMS23]. The largest class of unbounded rank known to satisfy the conjecture is Lawrence matroids, shown by Montejano and Ramírez-Alfonsín [MR15]. Furthermore, in [BBLP95] it is shown that for realizable oriented matroids of rank r on n elements, the number of complete cells is $2(n^{r-3}) + O(n^{r-4})$, and so Roudneff's conjecture holds asymptotically for realizable oriented matroids.

Another open problem on complete cells is attributed to McMullen in [Lar72] for the largest integer $\nu(r)$ such that that every uniform oriented matoids of rank r and $\nu(r)$ elements has a complete cell. The lower bound of $2r-1 \leqslant \nu(r)$ has been shown first for realizable oriented matroids in [Lar72] and then for general uniform oriented matroids [CdS85]. The common belief seems to be that this is tight. This has been shown for $r \leqslant 5$ [FLVS01], but remains open otherwise. After a series of results [Lar72, LV86a], the currently best-known upper bound $\nu(r) < 2(r-1) + \left\lceil \frac{r}{2} \right\rceil$ is due to Ramírez Alfonsín [Ram01].

A complete cell is analogous to a convex polytope, and this point of view connects complete cells to k-neighborly oriented matroids: see [HOKM23, Stu88a].

The tope graph To every simple oriented matroid M one can associate its *tope graph*, which is the subgraph of the hypercube Q_E on $\{+,-\}^E$ induced by the topes. It determines M uniquely up to isomorphism and therefore is an alternative point of view for the study of oriented matroids. The tope graph is a *partial cube*, i.e., an isometric subgraph of Q_E which by symmetry furthermore has a property called *antipodality*. This creates an important link to metric graph theory, in which partial cubes form one of the central classes. While in rank at most 3 tope graphs coincide with planar antipodal partial cubes [FH93], in general not all the antipodal partial cubes, a.k.a., *acycloids*, are tope graphs of oriented matroids. This has been known since Handa's work [Han87, Han93], and the question for a purely graph theoretical characterization of tope graphs [Han93, Problem 2] that can furthermore be verified in polynomial time [Fuk04, Problem 1.2] was around for a while. This has been accomplished in [KM20] and may be viewed as identifying the theory of oriented matroids with a part of metric graph theory. In particular many of the above problems can be conveniently stated in terms of tope graphs, and it will be exciting to see if tools from metric graph theory may shed light on longstanding questions.

3.7.2 Cocircuits

Problems on point configurations can generalize to problems on cocircuits in oriented matroids. As a charming example, we have the following: A natural open problem in oriented matroids is to find a generalization of the *Sylvester-Gallai Theorem*: for every set of points in the plane that does not lie on a single line there is a line that contains only two of the points. The question of how many such lines exist is the subject of conjectures of Dirac-Motzkin and Grünbaum and has been answered for large n by Green and Tao [GT13]. The theorem generalizes to oriented matroid of rank 3, where the quantitative question remains open [CS93]. Works of Hansen [Han65] and Shannon [Sha79] generalize the Sylvester-Gallai Theorem to realizable oriented matroids. Mandel pushed this further to oriented matroids with a general-position extension to a Euclidean oriented matroid [Man82]. The conjecture that the theorem holds for all oriented matroids is attributed to Murty by Mandel, who shows it to be equivalent to the statement that every orientable matroid has a disconnected hyperplane.

The *cocircuit graph* of an oriented matroid M can be interpreted as the 1-skeleton of the pseudosphere arrangement representing M. In contrast to the tope graph, the cocircuit graph does not uniquely determine an oriented matroid. Indeed, it does not even determine the number of nonloops of the oriented matroid [CFGdO00]. It is even unknown whether the cocircuit graph determines the rank of M. In contrast to tope graphs, no purely graph theoretic characterization is known for cocircuit graphs: see [KMBJJ14] for some ideas for sign-labeled graphs. Also, a polynomial time recognition algorithm is only available for the uniform case [FGK$^+$11, BFF01].

The oldest and most famous question in this context is whether the diameter of the cocircuit graph of an oriented matroid with n elements and rank r is bounded by $n-r+2$. This is known to hold for $r \leqslant 3$ [FGK$^+$11, BFF01], where it also has been related to the Hirsch conjecture. The survey article [ADLKZ21] reduced the above problem to the uniform case and answered the question affirmatively for $n \leqslant 9$. Indeed, while it is easy to see that the diameter of a cocircuit graph is $O(rn)$, a weaker question due to Fukuda asks whether the diameter of the cocircuit graph of an oriented matroid on n elements in $O(n)$ (independently of the rank).

3.8 Beyond oriented matroids

3.8.1 Complexes of oriented matroids

Where oriented matroids are abstractions of central hyperplane arrangements, *complexes of oriented matroids*, a.k.a. *conditional oriented matroids (COMs),* abstract intersections of arbitrary affine hyperplane arrangements with open polyhedra. In this *realizable* setting they appear for instance in neural codes [KLR23,IKR20] and in relation to the Varchenko-Gel'fand ring [DB23]. They were introduced by Bandelt, Chepoi, and Knauer [BCK18]. COMs capture several objects beyond oriented matroids, e.g., distributive latices and more generally median graphs a.k.a skeleta of CAT(0)-cube complexes [BC08, Gro87], linear extension graphs of posets [FM11, Naa00], and convex (semi)geometries [ES88] a.k.a (conditional) antimatroids [BCDK06, JW80]. The covector axioms of COMs are a relaxation of the oriented matroid covector axioms and make it clear that oriented matoids, affine oriented matroids, and lopsided sets a.k.a. ample systems [Law83] all find a common generalization in COMs. Other objects that can be expressed as COMs include

CAT(0)-Coxeter zonotopal complexes [HP98], Pasch and hypercellular graphs [CKM20].

Many fundamental notions such as minors, topes and tope graphs generalize to COMs, but also deeper results extend from oriented matroids to COMs, e.g., the factorization of the Varchenko determinant [HW19, HKK22, Ran21] and the characterization of tope graphs [KM20].

Some of the fundamental open problems for COMs are:

- **COMs from oriented matroids**: while any restriction [KM20] (a.k.a. supertope [HW19] a.k.a topal fiber [BCK18]) of an oriented matroid is a COM, it is open whether the converse holds. This is only known for affine oriented matroids which have an intrinsic axiomatization [BZ18] but has been conjectured in [BCK18, KM20, HKK22]. This is perhaps the central conjecture in the area, since it would yield many oriented matroid properties for COMs, e.g., a Topological Representation Theorem with pseudospheres and open pseudohemispheres.

- **weak maps**: a generalization of the fact that every oriented matroid is a weak map image of a uniform oriented matroid of the same rank has been conjectured in [CKP22], where the role of uniform oriented matroids is played by lopsided sets.

- **duality**: while lopsided sets as well as oriented matoids admit the notion of duality, for COMs no such notion is known, although definitions of circuits [DBPW22] and cocircuits [BCK18] have been proposed.

- **underlying unoriented theory**: while oriented matroids [Oxl92] and affine oriented matroids [Ard07] admit an underlying unoriented theory of (ordinary) matroids and semimatoids, no such class is known for COMs. A candidate is given by bouquets of matroids [CL89a, DL87, LD89].

- **sample compression**: through their topes COMs can be interpreted as set systems, and from this point of view they have been shown to satisfy the *labeled sample compression conjecture* [CKP23], one of the fundamental open problems in Computational Learning Theory: see [RR12] for more information. The stronger *unlabeled* version of the conjecture has been shown for oriented matroids and COMs admitting a corner peeling [Mar22].

- **beyond COMs**: COMs have been featured as perhaps *the* example of *CW left regular bands* in a work of Margolis, Saliola, and Steinberg [MSS21]. Other examples include the complex oriented matroids of Björner and Ziegler [Zie93b, BZ92]. As a consequence natural Markov chains such as the Tstetlin library can be generalized and analyzed as well as Euler formulas can be obtained. What properties of COMs can be extended to CW left regular bands?

3.8.2 Matroids over partial hyperstructures

Oriented matroids constitute one example of a theory of *matroids with extra structure*. Various attempts at other examples have been made over the years: a key difficulty in developing such a theory lies in finding suitable notions of duality and cryptomorphisms. Among the theories that succeeded at this to some extent are:

- *matroids over fuzzy rings*, see [Dre86b], [DW91], [DW92a]),
- *valuated matroids* [DW92b], [MS15], and
- *phased matroids* (also called *phirotopes* [BKRG03] and *complex matroids* [AD12]).

In 2016 Baker and Bowler generalized all of the above examples, as well as matroids and oriented matroids, into a comprehensive theory of *matroids over partial hyperstructures* [BB16], [BB19]. For an introduction, see [Bak17]. Some key examples:

- A field is a partial hyperstructure, and a rank r matroid over a field F on elements $[n]$ is a rank r subspace of F^n.
- The *sign hyperfield* $\mathbb{S} = \{0, +, -\}$ is a partial hyperstructure, and a matroid over $\mathbb{S}$ is an oriented matroid.
- The *Krasner hyperfield* $\mathbb{K} = \{0, \neq 0\}$ is a partial hyperstructure, and a matroid over $\mathbb{K}$ is an ordinary matroid.

A particularly intriguing aspect of the theory is its formulation of realizability. A morphism $\rho : H \to H'$ of partial hyperstructures induces a map $\hat{\rho}$ from matroids over H to matroids over H'. In particular,

- when H is a field and $H' = \mathbb{K}$ then the preimage under $\hat{\rho}$ of a matroid is its set of vector space realizations over H,
- when $H = \mathbb{S}$ and $H' = \mathbb{K}$ then the preimage under $\hat{\rho}$ of a matroid is its set of orientations, and
- when $H = \mathbb{R}$ and $H' = \mathbb{S}$ then the preimage under $\hat{\rho}$ of an oriented matroid is its realization space.

The *foundation* of a matroid M [BL] is a partial hyperstructure canonically associated to M that classifies representations of M over general hyperstructures. Baker and Lorscheid gave a presentation of the foundation of a matroid in terms of generators and relations, building on work of Tutte, Dress-Wenzel, and Gelfand-Rybnikov-Stone, which they used to prove results on realizations of matroids without large uniform minors. There is much room for further exploration of foundations.

Many of the questions on Grassmannians and flag spaces in Section 3.3 generalize to matroids over a partial hyperstructure with a compatible topology. A framework for this, with many open questions, is laid out in [AD19].

Acknowledgments

We thank Emeric Gioan, Arnau Padrol, Julian Pfeifle, Felipe Rincon, Manfred Scheucher, Thomas Zaslavsky.

4 Some Additions and Corrections.

In this section, we collect some notes, additions, corrections and updates to the 1993 book by Björner, Las Vergnas, Sturmfels, White & Ziegler [BLVS+99]. The list is far from complete (even in view of the points that we know about), and with your help we plan to expand it in the future.

Page 144, proof of Theorem 3.7.5

The reference to Proposition 3.7.3 should be to Proposition 3.7.2.

Page 148, Proposition 3.8.2

This result is actually due to da Silva ([dS87a], Chapter 6, Theorem 1).

Page 150, Section 3.9 "Historical Sketch"

Jaritz [Jar96, Jar97] gives a new axiomatic of oriented matroids in terms of "order functions" whose axioms and concepts she traces back to Sperner [Spe49] (1949!), Karzel [Kar69] etc. At the same time, Kalhoff [Kal00] reduces embedding questions about pseudoline arrangements, as solved by Goodman, Pollack, Wenger & Zamfirescu [JEGZ94, GPWZ96], back to 1967 results of Prieß-Crampe [PC67].

All this gets us closer to confirming the suspicion that probably Hilbert knew about oriented matroids...

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the same for a contraction that need not be simple. should be *the same for a contraction, which need not be simple.*

Page 176, proof of Proposition 4.3.1

There's a minor error in the proof. At the top of Page 176 the oriented matroid is assumed to be simple. Further down, in the argument (ii), an inductive argument assumes the same for a contraction that need not be simple. Perhaps the simplest fix is to drop the assumption of simplicity and talk about parallelism classes of nonloops rather than about elements.

Page 220, Exercise 4.28*.

Part (a) of this was already proved by Zaslavsky [Zas75b, Sect. 9]. However, part (b) remains open and should be an interesting challenge.

Page 227, Definition 5.1.3.

For condition (A2), if $S_A \cap S_e = S^{-1} = \emptyset$ is the empty sphere in a zero sphere $S_A \cong S^0$, then the sides of this empty sphere are the two points of S_A.

Page 244, Exercise 5.2(c).

Hochstättler [Hoc95] has shown that quite general arrangements of wild spheres also yield oriented matroids.

Page 270, Proposition 6.5.1.

Felsner [Fel97] has constructed a new and especially effective encoding scheme for wiring diagrams, which implies improved upper bound for the number of wiring diagrams and hence of simple pseudoline arrangements, namely

$$\log_2 s_n \ < \ 0.6988\, n^2.$$

Page 275:

Richter-Gebert [RG99] has proved (in 1996, and written up in 1998) that orientability is NP-complete [RG99]. (It's a beautiful paper!)

Page 279, Exercises 6.21(a)$^{(*)}$

The answer is "yes": this problem was solved in 1997, with an explicit construction, by Forge & Ramírez Alfonsín [FA98].

Page 289, proof of Theorem 7.1.8

The reference should be to Theorem 3.6.1*.

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, talking about a lexicographic extension $p = [e_1^{a_1}, \dots, e_k^{a_k}]$, it says that e_1 and p are inseparable, *covariant if* $a_1 = +$ *and contravariant if* $a_1 = -$. The last sentence must be *contravariant if* $a_1 = +$ *and covariant if* $a_1 = -$. Since $a_1 = +$ means that p and a_1 have the same sign in all cocircuits, and hence opposite in all circuits.

Page 334, Exercises 7.15(b)$^{(*)}$ and 7.17.

An explicit example of an oriented matroid that has a simple adjoint, but not a double adjoint was constructed by Hochstättler & Kromberg [HK96b, Kro95].

Also, they observed [HK96a, Kro95] that some assertions in Exercise 7.17 are not correct: Jürgen Richter-Gebert's [RG91, p. 117] 8-point torus is realizable over an ordered *skew field*, but not over $\mathbb{R}$. Therefore the oriented matroid given by such a skew realization has an infinite sequence of adjoints, but it is not realizable in $\mathbb{R}^4$.

Page 337, Exercises 7.44*.

No one seems to remember the example: so consider this to be an open problem. (The non-existence of such an example is also discussed, as a Conjecture of Brylawski, in McNulty [McN94].)

Page 385, McMullen's problem on projective transformations.

Forge, Schuchert, and Las Vergnas [FLVS01] have found a configuration of 10 points in general position in affine 4-space that no projective transformation can put into convex position. This solves McMullen's problem for $d = 4$ resp. $r = 5$ with $f(4) = g(5) = 9$.

This is consistent with the conjecture that the inequalities $2d + 1 \leqslant g(d + 1) \leqslant f(d)$ [sic.!] hold with equality also for $d > 4$.

Page 396.

Proposition 9.4.2 is true only for $n \geqslant r+2$. For $n = r+1$ the matroid is one single circuit, the inseparability graph is a complete graph, etc.

Page 405 (top).

It is not true that the sphere $\mathcal{S} = M^9_{963}$ is neighborly: the edges 13 and 24 are missing (in the labeling used in [BLVS$^+$99]). Thus Shemer's Theorem 9.4.13 cannot be applied here. A proof that the sphere admits at most one matroid polytope, `AB`(9), was given by Bokowski [Bok] in 1978 (see also Altshuler, Bokowski & Steinberg [ABS80] and Antonin [Ant82]). It is described in detail in Bokowski & Schuchert [BS95a]. (The oriented matroid `RS`(8) discussed in [BLVS$^+$99, Sect. 1.5] arises as a contraction of the oriented matroid `AB`(9).)

Page 413, Exercise 9.12$^{(*)}$.

Bokowski & Schuchert [BS95a] showed that the smallest example (both in terms of its rank $r = 5$ and in terms of its number of vertices $n = 9$), is given by Altshuler's sphere M^9_{963}.

Page 424.

In Definition 10.1.8, delete "infeasible oriented matroid program" resp. "unbounded oriented matroid program."

After this, the cocircuit Y should be $Y = (00{+}{+}{+}|{+}{-})$, the circuit X should be $X = (0{+}00{+}|{-}{+})$, and the circuit X_0 should be $X^0 = (000{+}{+}|{-}{+})$

Page 426, Proof of Corollary 10.1.10.

"Orthogonality of circuits and **co**circuits"

Page 481, Definition A.1.1

As noted in Section 3.1 here, there's an omission in this definition.

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